Pismeni ispit iz Diferencijalne geometrije, 05.07.2013. (ispit pisati isključivo hemiskom olovkom plave ili crne tinte, obavezno navesti formulu koju koristite i značenje simbola iz napisane formule)

1. Data je kriva

$$L : x = a\cos^2 u,\ \ y = a\sin u\cos u,\ \ z = a\sin u$$

(a) Pokazati da kriva L leži u presjeku jedne lopte i cilindra čija je generatrisa paralelna osi $0z$ i odrediti jednačine tih površi.

(b) Odrediti jednačinu oskulatorne ravni krive L za $u = \dfrac{\pi}{2}$.

2. Data je kriva

$$L : y = \frac{1}{2m}x^2,\ \ z = \frac{1}{6m^2}x^3$$

gdje je m parametar nezavisan od x i y. Naći ortove prirodnog triedra, fleksiju i torziju krive L u tački za koju je $x = 2m$.

3. Površ Γ definisana je vektorskom jednačinom

$$\vec{r} = (u\sin v,\ u\cos v,\ v).$$

Na površi je zadan krivoliniski trougao

$$0 \le u \le \operatorname{sh} v,\ \ 0 \le v \le v_0.$$

Izračunati uglove trougla.

4. Površ Γ definisana je vektorskom jednačinom

$$\vec{r} = (u\sin v,\ u\cos v,\ v).$$

(a) Naći prvu kvadratnu formu površi.

(b) Na površi je zadan krivoliniski trougao

$$0 \le u \le \operatorname{sh} v,\ \ 0 \le v \le v_0.$$

Izračunati površinu i dužine strana trougla.

Zadaci su skinuti sa stranice pf.unze.ba/nabokov.
Za uočene greške pisati na *infoarrt@gmail.com*

(#) Data je kriva

$$L: \; x = a\cos^2 u, \quad y = a\sin u\cos u, \quad z = a\sin u$$

a) Pokazati da kriva L leži u presjeku jedne lopte i cilindra čija je generatrisa paralelna osi Oz i odrediti jednačine tih površi.

b) Odrediti jednačinu oskulatorne ravni krive L za $u = \frac{\pi}{2}$.

Rj. a) Primjetimo da je

$$x^2 + y^2 + z^2 = a^2\cos^4 u + a^2\sin^2 u\cos^2 u + a^2\sin^2 u =$$

$$= a^2\cos^2 u\underbrace{(\cos^2 u + \sin^2 u)}_{=1} + a^2\sin^2 u =$$

$$= a^2(\cos^2 u + \sin^2 u) = a^2$$

tj. $x^2 + y^2 + z^2 = a^2$ – jednačina sfere

i

$$x^2 + y^2 = a^2\cos^4 u + a^2\sin^2 u\cos^2 u = a^2\cos^2 u(\cos^2 u + \sin^2 u)$$

$$= a^2\cos^2 u = a \cdot \underbrace{a\cos^2 u}_{x} = ax$$

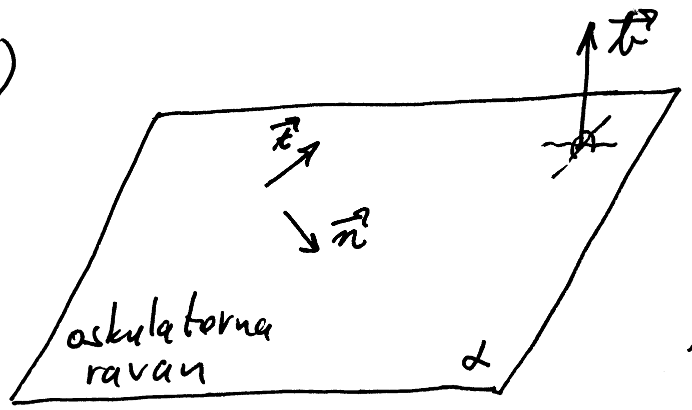
tj.

$x^2 + y^2 = ax$ – jednačina cilindrične površi čija je generatrisa paralelna osi Oz.

Vidimo da je kriva L određena presjekom ove dvije površi.

$$L: \begin{cases} x^2+y^2+z^2=a^2 \\ x^2+y^2=ax \end{cases}$$

b)



$$\vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}}$$

$$\dot{\vec{r}} = \{-2a\sin u\cos u,\ a(\cos^2 u-\sin^2 u),\ a\cos u\}$$

$$\ddot{\vec{r}} = \{-2a\cos 2u,\ -2a\sin 2u,\ -a\sin u\}$$

Za $u=\frac{\pi}{2}$ je

$$\left.\begin{array}{l} \dot{\vec{r}} = (0,-a,0) \\ \ddot{\vec{r}} = (2a,0,-a) \end{array}\right\} \Rightarrow \dot{\vec{r}} \times \ddot{\vec{r}} =$$

$$\vec{r} = (0,0,a)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -a & 0 \\ 2a & 0 & -a \end{vmatrix} =$$

$$= (a^2, 0, 2a^2)$$

Vidimo da je $M(0,0,a)$ proizvoljna tačka krive

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

$$a^2(x-0)+0\cdot(y-0)+2a^2(z-a)=0 \qquad /:a^2$$

$$x+2z-2a=0$$

je tražena jednačina oskulatorne ravni

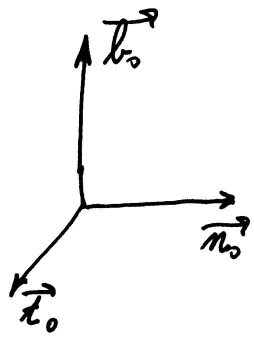
Data je kriva

$$L: \ y = \frac{1}{2m} x^2, \quad z = \frac{1}{6m^2} x^3$$

gdje je m parametar nezavisan od x i y. Nađi ortove prirodnog triedra, fleksiju i torziju krive L u tački za koju je $x = 2m$.

Rj. Jednačina krive L u vektorskom obliku glasi

$$\vec{r} = \left\{ t, \frac{1}{2m} t^2, \frac{1}{6m^2} t^3 \right\}$$



prirodni triedar

Znamo da vrijedi

$$\vec{t} = \dot{\vec{r}}$$

$$\vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}}$$

$$\vec{n} = \vec{b} \times \vec{t}$$

Ortovi prirodnog triedra su određeni relacijama $\vec{t}_0 = \frac{\vec{t}}{|\vec{t}|}, \ \vec{b}_0 = \frac{\vec{b}}{|\vec{b}|}, \ \vec{n}_0 = \frac{\vec{n}}{|\vec{n}|}$

$$\dot{\vec{r}} = \left\{ 1, \frac{1}{m} t, \frac{1}{2m^2} t^2 \right\}$$

$$\ddot{\vec{r}} = \left\{ 0, \frac{1}{m}, \frac{t}{m^2} \right\} \qquad \dddot{\vec{r}} = \left\{ 0, 0, \frac{1}{m^2} \right\}$$

Za $x = 2m$ je

$$\dot{\vec{r}} = \{1, 2, 2\},$$

$$\ddot{\vec{r}} = \left\{ 0, \frac{1}{m}, \frac{2}{m} \right\}$$

$$\dddot{\vec{r}} = \left\{ 0, 0, \frac{1}{m^2} \right\}$$

pa je $\vec{t} = (1,2,2)$, $|\vec{t}| = \sqrt{9} = 3$

$$\vec{b} = \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ 0 & \frac{1}{m} & \frac{2}{m} \end{vmatrix} = \left(\frac{2}{m}, \frac{-2}{m}, \frac{1}{m}\right)$$

$$|\vec{b}| = \sqrt{\frac{9}{m^2}} = \frac{3}{|m|}$$

$$\vec{n} = \vec{b} \times \vec{t} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{2}{m} & -\frac{2}{m} & \frac{1}{m} \\ 1 & 2 & 2 \end{vmatrix} = \left(-\frac{6}{m}, -\frac{3}{m}, \frac{6}{m}\right)$$

$$|\vec{n}| = \sqrt{\frac{81}{m^2}} = \frac{9}{|m|}$$

Prema tome

$$\vec{t} = \frac{1}{3}(\vec{i} + 2\vec{j} + 2\vec{k}), \quad \vec{b} = \frac{1}{3}(2\vec{i} - 2\vec{j} + \vec{k}), \quad \vec{n} = \frac{1}{3}(-2\vec{i} - \vec{j} + 2\vec{k})$$

Fleksiju i torziju možemo izračunati po obrascima

$$K = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \qquad \tau = -\frac{1}{T} = \frac{\dot{\vec{r}}(\ddot{\vec{r}} \times \dddot{\vec{r}})}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \frac{3}{|m|}, \quad |\dot{\vec{r}}| = 3, \quad |\dot{\vec{r}}|^3 = 27 \quad \Rightarrow \quad K = \frac{1}{9|m|}.$$

$$\dot{\vec{r}}(\ddot{\vec{r}} \times \dddot{\vec{r}}) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & \frac{1}{m} & \frac{2}{m} \\ 0 & 0 & \frac{1}{m^2} \end{vmatrix} = 1 \cdot \frac{1}{m} \cdot \frac{1}{m^2} = \frac{1}{m^3}, \quad |\dot{\vec{r}} \times \ddot{\vec{r}}|^2 = \frac{9}{m^2}$$

$$\tau = \frac{1}{9m}$$

(#) Površ Π definisana je vektorskom jednačinom

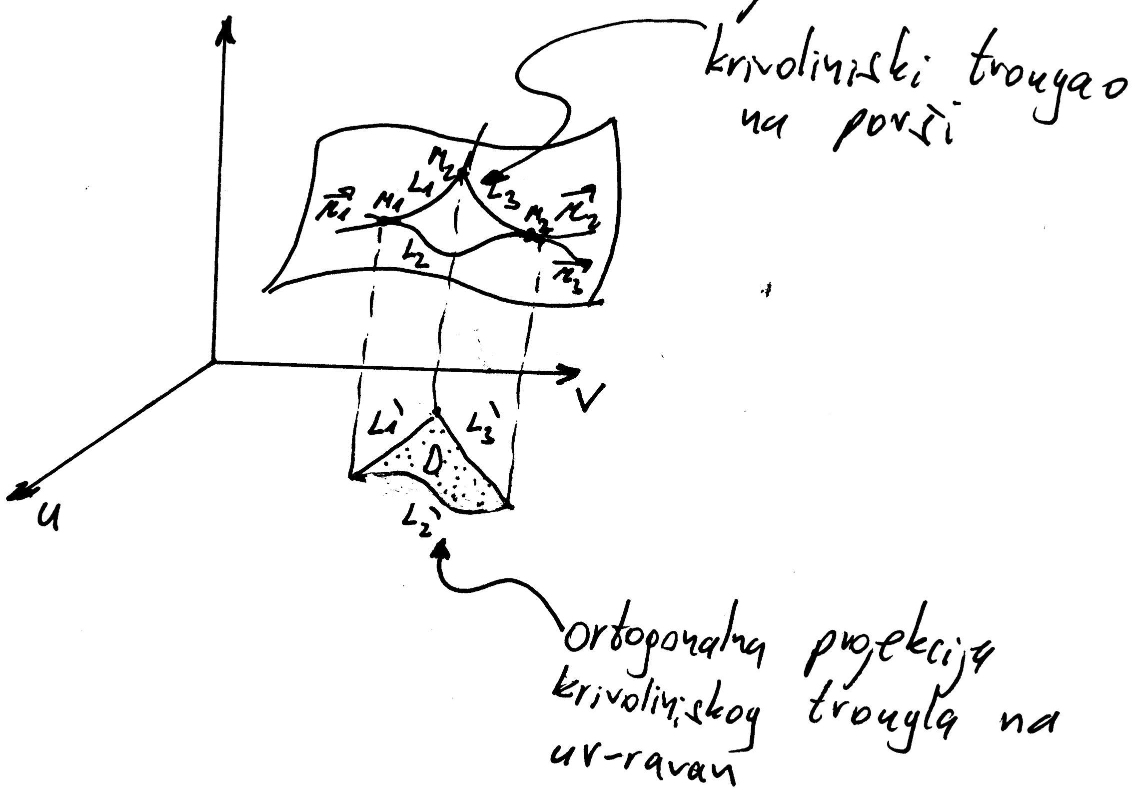
$$\vec{r} = (u\sin v,\ u\cos v,\ v).$$

Na površi je zadan krivolinijski trougao

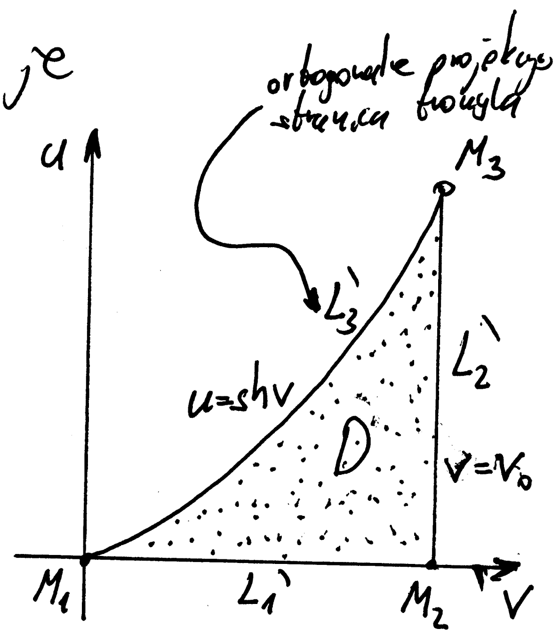
$$0 \le u \le \operatorname{sh} v, \quad 0 \le v \le v_0.$$

Izračunati uglove trougla.

Rj. Posmatrajmo neku površ u opštem slučaju



U našem slučaju ortogonalna projekcija je



Odredimo jednačine stranica krivolinijskog trougla.

$$L_1' : u = 0 \xrightarrow{0 \le v \le v_0}$$

$$L_1 : \vec{r}_1 = (0, 0, v), \quad 0 \le v \le v_0 ,$$

L_2': $v=v_0$, $0\le u\le \operatorname{sh}v_0 \Rightarrow$

$$\Rightarrow \quad L_2: \vec{r}_2=(u\sin v_0,\ u\cos v_0,\ v_0),\quad 0\le u\le \operatorname{sh}v_0$$

L_3': $u=\operatorname{sh}v$, $0\le v\le v_0 \Rightarrow$

$$L_3: \vec{r}_3=(\operatorname{sh}v\sin v,\ \operatorname{sh}v\cos v,\ v),\quad 0\le v\le v_0$$

Odredimo presječne tačke krivih $\vec{r}_1, \vec{r}_2$ i $\vec{r}_3$

$$\left.\begin{matrix}\vec{r}_1=(0,0,v)\\ \vec{r}_2=(u\sin v_0,\ u\cos v_0,\ v_0)\end{matrix}\right\} \Rightarrow \begin{matrix}u\sin v_0=0\\ u\cos v_0=0\\ v_0=v\end{matrix} \Rightarrow M_1(\overset{v}{v_0},\overset{u}{0})$$

$$\left.\begin{matrix}\vec{r}_1=(0,0,v)\\ \vec{r}_3=(\operatorname{sh}v\sin v,\ \operatorname{sh}v\cos v,\ v)\end{matrix}\right\} \Rightarrow \begin{matrix}\operatorname{sh}v\sin v=0\\ \operatorname{sh}v\cos v=0\end{matrix} \Rightarrow M_2(\overset{v}{0},\overset{u}{0})$$

slično za $\vec{r}_2$ i $\vec{r}_3$ $\quad M_3(\overset{v}{v_0},\overset{u}{\operatorname{sh}v_0})$

<u>Ugao između dvije krive linije se definiše kao ugao između tangenti u presječnoj tački tih krivih.</u>

$\dot{\vec{r}}_1=(0,0,1)$

$\dot{\vec{r}}_2=(\sin v_0,\ \cos v_0,\ 0)$

$\dot{\vec{r}}_3=(\operatorname{ch}v\sin v+\operatorname{sh}v\cos v,\ \operatorname{ch}v\cos v-\operatorname{sh}v\sin v,\ 1)$

Znamo da je

$$\vec{x}\cdot\vec{y}=|\vec{x}||\vec{y}|\cos\measuredangle(\vec{x},\vec{y})$$

$$\cos\measuredangle(\vec{x},\vec{y})=\frac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|}$$

Time je

$$\cos\varphi_1=\left.\frac{\dot{\vec{r}}_1\cdot\dot{\vec{r}}_2}{|\dot{\vec{r}}_1||\dot{\vec{r}}_2|}\right|_{M_1}=\overset{\text{ZA VJEŽBU}}{\ldots}=0 \Rightarrow \varphi_1=\frac{\pi}{2}$$

$$\cos\varphi_2=\cos\varphi_3=\ldots=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2} \Rightarrow \varphi_2=\varphi_2=\frac{\pi}{4}$$

traženi uglovi

\# Površ Γ definisana je vektorskom jednačinom

$$\vec{r} = (u \sin v,\ u \cos v,\ v)$$

a) Naći prvu kvadratnu formu površi.

b) Na površi je zadan krivolinijski trougao

$$0 \le u \le \operatorname{sh} v, \quad 0 \le v \le v_0$$

Izračunati površinu, dužine strana i uglove trougla.

Rj:

a) Prva kvadratna forma površi je

$$F_1 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

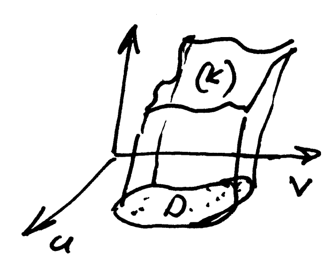
$$E = (\vec{r}\,'_u)^2, \quad F = (\vec{r}\,'_u \cdot \vec{r}\,'_v), \quad G = (\vec{r}\,'_v)^2$$

$$\left.\begin{array}{l} \vec{r}\,'_u = (\sin v,\ \cos v,\ 0) \\ \vec{r}\,'_v = (u \cos v,\ -u \sin v,\ 1) \end{array}\right\} \Rightarrow \quad \begin{array}{l} E = 1 \\ F = 0 \\ G = 1 + u^2 \end{array}$$

Prva kvadratna forma je

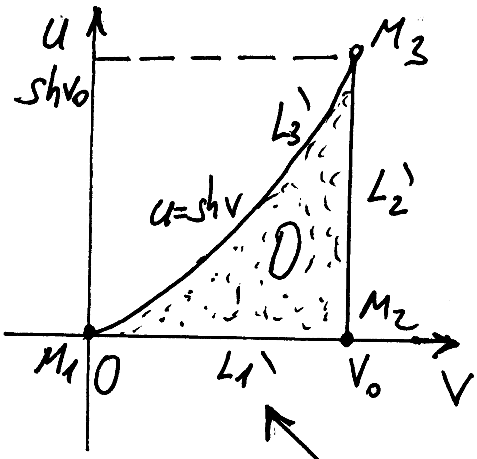
$$F_1 = ds^2 = du^2 + (1+u^2)\,dv^2$$

b) Površinu dijela površi računamo po formuli:



$$P = \iint\limits_{(K)} dS = \iint\limits_{D} \sqrt{EG - F^2}\,du\,dv$$

Oblast D nije teško skicirati



ortogonalne projekcije stranica trougla

$$D: \begin{cases} 0 \leq u \leq \operatorname{sh} v \\ 0 \leq v \leq v_0 \end{cases}$$

U našem slučaju je

$$P = \iint_D \sqrt{1+u^2}\, du\, dv = \int_0^{v_0} dv \int_0^{\operatorname{sh} v} \sqrt{1+u^2}\, du \overset{(*)}{=}$$

$$\int \sqrt{1+u^2}\, du = \underset{\dots}{\text{ZA VJEŽBU}} = \frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)$$

$$\int_0^{\operatorname{sh} v} \sqrt{1+u^2}\, du = \frac{u}{2}\sqrt{1+u^2}\Big|_0^{\operatorname{sh} v} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\Big|_0^{\operatorname{sh} v} =$$

$$= \frac{1}{2}\operatorname{sh} v\sqrt{1+\operatorname{sh}^2 v} + \frac{1}{2}\ln\left(\operatorname{sh} v + \sqrt{1+\operatorname{sh}^2 v}\right) =$$

(Znamo da je $1+\operatorname{sh}^2 v = \operatorname{ch}^2 v$)

$\operatorname{ch} v \geq 0 \quad \forall v \in \mathbb{R}$

$$= \frac{1}{2}\operatorname{sh} v\,|\operatorname{ch} v| + \frac{1}{2}\ln(\operatorname{sh} v + \operatorname{ch} v)$$

$$= \frac{1}{2}\operatorname{sh} v \cdot \operatorname{ch} v + \frac{1}{2}\ln(\operatorname{sh} v + \operatorname{ch} v)$$

$$\overset{(*)}{=} \int_0^{v_0}\left(\frac{1}{2}\operatorname{sh} v \operatorname{ch} v + \frac{1}{2}\ln(\operatorname{sh} v + \operatorname{ch} v)\right) dv = \underbrace{\frac{1}{2}\int_0^{v_0} \operatorname{sh} v \operatorname{ch} v\, dv}_{I_1} + \underbrace{\frac{1}{2}\int_0^{v_0} \ln(\operatorname{sh} v + \operatorname{ch} v)\, dv}_{I_2}$$

$$I_1 = \frac{1}{2}\int_0^{v_0} \operatorname{sh} v \operatorname{ch} v\, dv = \frac{1}{2}\int_0^{v_0} \operatorname{sh} v\, d(\operatorname{sh} v) = \frac{1}{2}\cdot\frac{1}{2}\operatorname{sh}^2 v\Big|_0^{v_0} = \frac{1}{4}\operatorname{sh}^2 v$$

$$\operatorname{sh} x + \operatorname{ch} x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$$

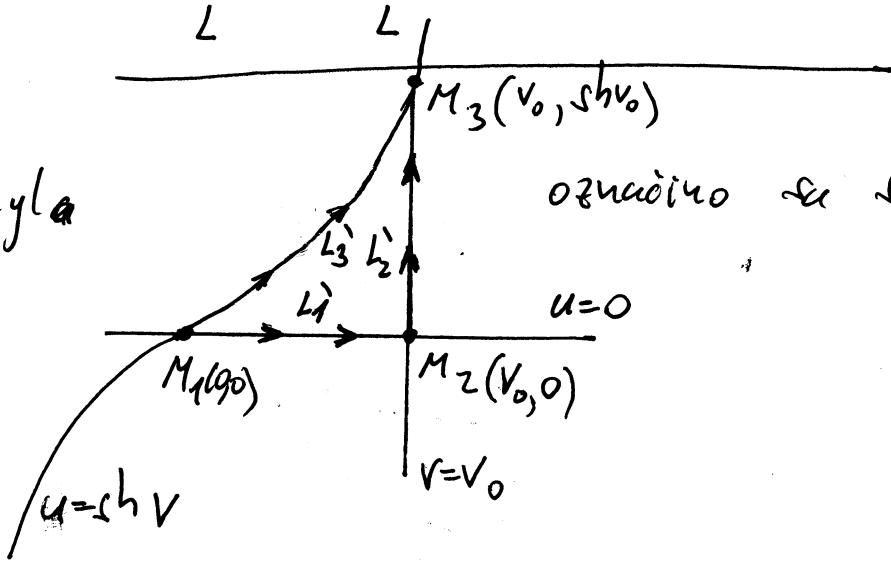
$$I_2 = \frac{1}{2}\int_0^{v_0} \ln(e^v)\,dv = \frac{1}{2}\int_0^{v_0} v\,dv = \frac{1}{4}v^2\Big|_0^{v_0} = \frac{1}{4}v_0^2$$

Prema tome $$P = \frac{1}{4}\left(v_0^2 + \mathrm{sh}^2 v_0\right)$$ tražena površina

Dužina luka L se računa po obrascu

$$s = \int_L ds = \int_L \sqrt{du^2 + (1+u^2)\,dv^2}$$

Dužine stranica trougla označimo sa s_1, s_2 i s_3



$$s_1 = \int_{L_1} ds = \int_{L_1} \sqrt{du^2 + (1+u^2)dv^2} = \int_0^{v_0} dv = v_0$$

$$s_2 = \int_{L_2} \sqrt{du^2 + (1+u^2)\,dv^2} = \left| \begin{matrix} L_2: v = v_0 \\ dv = 0 \end{matrix} \right| = \int_0^{\mathrm{sh} v_0} du = \mathrm{sh}\, v_0$$

$$s_3 = \int_{L_3} \sqrt{du^2 + (1+u^2)dv^2} = \left| \begin{matrix} u = \mathrm{sh}\, v \\ du = \mathrm{ch}\, v\, dv \\ du^2 = \mathrm{ch}^2 v\, dv^2 \end{matrix} \right| = \int_0^{v_0} \sqrt{\mathrm{ch}^2 v + 1 + \mathrm{sh}^2 v}\, dv =$$

$$= \left| \begin{matrix} \text{znamo da} \\ 1+sh^2 v = ch^2 v \end{matrix} \right| = \sqrt{2} \int_0^{v_0} ch\, v \, dv = \sqrt{2}\, sh\, v_0$$

Dužine strana krivoliniskog trougla na datoj površini su

$$s_1 = v_0, \quad s_2 = sh\, v_0, \quad s_3 = \sqrt{2}\, sh\, v_0$$

Jednačine stranica krivoliniskog trougla su

L_1: $\vec{r}_1 = (0, 0, v)$, $0 \leq v \leq v_0$

L_2: $\vec{r}_2 = (u \sin v_0, u \cos v_0, v_0)$, $0 \leq u \leq sh\, v_0$

L_3: $\vec{r}_3 = (sh\, v \sin v, sh\, v \cos v, v)$, $0 \leq v \leq v_0$

Presječna tačka stranica L_1 i L_2 je $M_1(v_0, 0)$, stranica L_1 i L_3 je $M_2(0,0)$ i L_2 i L_3 je $M_3(v_0, sh\, v_0)$

Stoga je
$$\cos\varphi_1 = \frac{\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2}{|\dot{\vec{r}}_1||\dot{\vec{r}}_2|}\Bigg|_{M_1} \overset{\text{za vježbu}}{=} 0$$

$$\cos\varphi_2 = \frac{\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_3}{|\dot{\vec{r}}_1||\dot{\vec{r}}_3|}\Bigg|_{M_2} = \ldots^{\text{za vježbu}} = \frac{1}{\sqrt{2}} = \frac{\dot{\vec{r}}_2 \cdot \dot{\vec{r}}_3}{|\dot{\vec{r}}_2| \cdot |\dot{\vec{r}}_3|}\Bigg|_{M_3} = \cos\varphi_3$$

i $\varphi_1 = \frac{\pi}{2}$, $\varphi_2 = \varphi_3 = \frac{\pi}{4}$ traženi uglovi